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1991 J. Phys. A: Math. Gen. 24 2847

(http://iopscience.iop.org/0305-4470/24/12/023)

J Phys A. Math Gen 24 (1991) 2847–2861 Printed in the UK

The *XXZ* model with Beraha values

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Received 9 November 1990, in final form 15 March 1991

Abstract. In this paper we study an *XXZ* model of spin-$\frac{1}{2}$ chains with the quantum sl(2) symmetry, where we reduce the Hamiltonian equation to a set of coupled linear algebraic equations. When q is a root of unity, some eigenstates of the Hamiltonian coincide with each other, and a new state is obtained through a limit process from their difference. This eigenspace corresponds to a reducible but indecomposable representation (type I) of the quantum sl(2) enveloping algebra

1. Introduction

Group theory is a powerful tool for studying the symmetry properties of systems. The *XXX* model of spin-$\frac{1}{2}$ chains has su(2) symmetry, and the eigenstates are the direct products of the spin states combined with the Young operators. There is an *XXZ* model of spin-$\frac{1}{2}$ chains [1] with the symmetry of the quantum sl(2) universal enveloping algebra (U_q sl(2)). The properties of this system can be studied by quantum Young operators.

The *XXZ* model was studied in terms of the Bethe ansatz 25 years ago [2]. Recently, Alcaraz *et al* [1] studied the *XXZ* model with a boundary term so that it has the symmetry U_q sl(2). In this case, the eigenenergies and eigenstates can be expressed by the impulsions k_j, which satisfy the compatibility conditions although they are hard to solve analytically. This method is not convenient enough for discussing the cases where q is a root of unity*. Pasquier and Saleur [4] pointed out that the eigenstates obtained by the Bethe ansatz are the highest weights of the irreducible representations of U_q sl(2). They discussed the properties of the representations of U_q sl(2) for q being a root of unity where some representations become reducible but indecomposable (type I). However, they did not construct the eigenstates for these q-values explicitly.

Recently, Levy [5] constructed the quantum Young operators by Temperley Lieb algebras and sketched the method for solving the Hamiltonian equation of the *XXZ* spin-$\frac{1}{2}$ chain model, both for generic and Beraha q-values. He used a simple example

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* When $q = \exp(ip'\pi/p)$, $q+q^{-1} = 2\cos(p'\pi/p)$, where p and p' are integers. Usually, $(2\cos\pi/n)^2$ is called the Beraha value [1, 3] Hereafter, if there is no confusion, we call the q-value which is a root of unity the Beraha q-value.

0305-4470/91/122847+15$03.50

to show the method but omitted the details. This is a typical and very interesting method of applying the quantum group to physical problems.

In the present paper, we will develop this method, including the following:

(i) To obtain the explicit forms of the Young operators where a factor Z_n^N corresponding to the one row part of the Young pattern was not given by Levy explicitly.

(ii) To reduce the Hamiltonian equation into a set of coupled linear algebraic equations, that is, the problem to solve the spin-$\frac{1}{2}$ chain model becomes the usual eigenvalue problem of a matrix with finite dimensions. The eigenvalues of the Hamiltonian are known from the Bethe ansatz.

(iii) When q is a root of unity, some eigenstates coincide with each other and the representation of the degenerate energy becomes reducible but indecomposable (type I). We will show that the coincidental states must have a zero norm, and a new independent state can be obtained by an appropriate limit process to complement the invariant space corresponding to the type I representation.

(iv) When q is a root of unity, Pasquier and Saleur [4] pointed out the condition for type I representations from the Casimir operator and the quantum dimension of the representation. Levy [5] also gave the condition without proof. We will show that it can also be obtained from the compatibility conditions of the impulsions.

The plan of this paper is as follows. In section 2 we sketch how to construct quantum Young operators from Temperley Lieb algebras. The factor Z_n^N corresponding to the one row part of the Young pattern is constructed explicitly. The quantum Young operators are used to combine the product of the spin states to belong to an irreducible representation of $U_q\,\mathrm{sl}(2)$. Then, in terms of the multiplication rules of the Temperley Lieb algebras, the general rules for reducing the Hamiltonian equation to a set of coupled linear algebraic equations are obtained in section 3. Through two simple examples the relations between this method and those of the Bethe ansatz are shown in section 4. When q is a root of unity, some energies corresponding to different representations may coincide with each other so that a reducible but indecomposable representation appears. In section 5 we will discuss the condition for appearance of type I representations from the energy formula. The new independent eigenstates to complement the invariant space corresponding to the type I representations will also be calculated there through an appropriate limit process.

2. The quantum Young operator

The spectrum-independent solution to the Yang–Baxter equation for the fundamental representation of $U_q\,\mathrm{sl}(2)$ is

$$\check{R}_q=\begin{pmatrix}1&0&0&0\\0&0&q&0\\0&q&1-q^2&0\\0&0&0&1\end{pmatrix} \tag{1}$$

which satisfies the reduction relation

$$(\check{R}_q-1)(\check{R}_q+q^2)=0. \tag{2}$$

When $q=1$, $\check{R}_q$ becomes the transposition P. The monodromy representation of the braid group B_{N+1} can be constructed from $\check{R}_q$:

$$b_i=\mathbf{1}^{(1)}\otimes\ldots\otimes\mathbf{1}^{(i-1)}\otimes\check{R}_q\otimes\mathbf{1}^{(i+2)}\otimes\ldots\otimes\mathbf{1}^{(N+1)}. \tag{3}$$

Define

$$e_i = q^{-1}(\mathbf{1} - b_i)$$
$$= \mathbf{1}^{(1)} \otimes \ldots \otimes \mathbf{1}^{(i-1)} \otimes E \otimes \mathbf{1}^{(i+2)} \otimes \ldots \otimes \mathbf{1}^{(N+1)} \tag{4a}$$

where

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= -\frac{1}{2}\left(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \frac{[2]}{2} \sigma^z \otimes \sigma^z\right)$$
$$+ \frac{[2]}{4} \mathbf{1} - \frac{q - q^{-1}}{4} (\sigma^z \otimes \mathbf{1} - \mathbf{1} \otimes \sigma^z) \tag{4b}$$

e_i is the quantum analogue of $\mathbf{1} - P_{i(i+1)}$, where $P_{i(i+1)}$ is the transposition of the ith and the $(i+1)$th spaces. Obviously, e_i, $1 \leq i \leq N$, satisfies the Temperley Lieb algebra A_{Nq}:

$$e_i^2 = [2] e_i$$
$$e_i e_{i\pm 1} e_i = e_i \tag{5}$$
$$e_i e_j = e_j e_i \qquad \text{if } |i - j| \geq 2$$

where

$$[m] \equiv \frac{q^m - q^{-m}}{q - q^{-1}} \tag{6}$$
$$[m]! = [m][m-1] \ldots [2][1].$$

From (2) we have

$$e_i b_i = b_i e_i = -q^2 e_i. \tag{7}$$

Owing to the commutability between the quantum group and the braid group, e_i is commutable with the coproduct of $U_q\,\mathrm{sl}(2)$

$$[e_i, \Delta(s_a)] = 0 \qquad a = z, + \text{ and } - \tag{8}$$

where

$$\Delta(s_z) = \sum_{l=1}^{N+1} \mathbf{1}^{(1)} \otimes \ldots \otimes \mathbf{1}^{(l-1)} \otimes s_z \otimes \mathbf{1}^{(l+1)} \otimes \ldots \otimes \mathbf{1}^{(N+1)}$$
$$\Delta(s_\pm) = \sum_{l=1}^{N+1} q^{s_z} \otimes \ldots \otimes q^{s_z} \otimes s_\pm^{(l)} \otimes q^{-s_z} \otimes \ldots \otimes q^{-s_z} \tag{9}$$

where $s_\pm^{(l)}$ is located at the lth spin space, and

$$s_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

For U_q sl(2) only the two-row Young pattern $(N-n+1, n)$ is relevant, where $(N+1)$ is the length of the spin chains. The box number n in the second row, $n \leqslant (N+1)/2$, is related to the number of the down spins of the highest weight, as shown in the next section, and $(n-1)$ is called the jump number [5]. For each Young pattern we only need one explicit form of the quantum Young operator, corresponding to the following Young tableau [5]:

1	3	...	$2n-1$	$2n+1$	...	$N+1$
2	4	...	$2n$			

$$\begin{aligned} &Y_n = S_n Z_n^N = Z_n^N S_n \\ &S_n = e_1 e_3 \ldots e_{2n-1} \\ &e_j Z_n^N = Z_n^N e_j = 0 \qquad \text{if } j > 2n \end{aligned} \tag{11}$$

and Z_n^N contains $e_{2n+1}, \ldots, e_N$. The explicit form of Z_n^N can be constructed by $g_i(u)$ [6]:

$$\begin{aligned} &g_i(u) = [1+u] - [u]e_i \\ &Z_n^{2n} = 1 \qquad Z_n^{2n+1} = g_{2n+1}(1) \\ &Z_n^{2n+2} = Z_n^{2n+1} g_{2n+2}(2) Z_n^{2n+1} \\ &\qquad \ldots \\ &Z_n^N = Z_n^{N-1} g_N(N-2n) Z_n^{N-1}. \end{aligned} \tag{12}$$

It is straightforward to prove

$$\begin{aligned} &Z_n^N Z_n^N \propto Z_n^N \\ &S_n w S_n \propto S_n \qquad \forall w \in A_{(2n)q} \\ &Y_n w Y_n \propto Y_n \qquad \forall w \in A_{Nq} \\ &Y_n Y_n = a Y_n \qquad a \neq 0 \end{aligned} \tag{13}$$

Therefore, Y_n is a primitive idempotent, and wY_n, $w \in A_{Nq}$, is a primitive left ideal. The independent bases in the left ideal are ${}^nC_{(m)} Z_n^N$:

$$\begin{aligned} &{}^nC_{(m)} \equiv {}^nC_{m_1 m_2 \ldots m_n} \equiv C_1^{m_1} C_3^{m_2} \ldots C_{2n-1}^{m_n} \\ &1 \leqslant m_1 < m_2 < \ldots < m_n \leqslant N, m_i \geqslant (2i-1) \\ &C_i^m = e_m e_{m-1} \ldots e_i. \end{aligned} \tag{14}$$

3. *XXZ* model of spin-$\frac{1}{2}$ chains with U_q sl(2) symmetry

Consider an *XXZ* model of $(N+1)$ spin chains with the Hamiltonian

$$\begin{aligned} H &= \sum_{i=1}^{N} e_i \\ &= -\frac{1}{2} \sum_{i=1}^{N} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{[2]}{2} \sigma_i^z \sigma_{i+1}^z \right) \\ &\quad + \frac{[2]}{4} N1 - \frac{q-q^{-1}}{4} (\sigma_1^z - \sigma_{N+1}^z) \end{aligned} \tag{15}$$

H is symmetric and the energies are real if $q^* = q^{-1}$ [1]. From (8) the Hamiltonian has U_q sl(2) symmetry. At first, we assume q is not a root of unity. Each eigenspace corresponds to an irreducible representation denoted by a Young pattern or n. On the other hand, for a given Young pattern, there are several independent spaces obtained from the direct products of the spin states applied by the Young operators ${}^nC_{(m)}Z_n^N$. The eigenstates are their appropriate combinations. Notice that the quantum Young operators are commutable with the coproducts $\Delta(s_a)$. It is obvious from (4) that the number of down spins of a state which are not annihilated by the Young operator ${}^nC_{(m)}Z_n^N$ must be not less than n, and a state of the highest weight of the representation has n down spins located in the first $2n$ positions. Denote a state with n down spins by $|x_1, x_2, \ldots, x_n\rangle$, $1 \leq x_1 < \ldots < x_n \leq N+1$, where x_is denote the locations of the down spins on the chain. For definiteness we define the state with the highest weight as follows:

$$ {}^n\Psi_{(m)j} \equiv {}^nC_{(m)}|1\ 3\ 5\ \ldots(2n-1)\rangle \qquad j = \tfrac{1}{2}(N-2n+1) \tag{16} $$

where the constant factor obtained by applying Z_n^N on the state $|1\ 3 \ldots (2n-1)\rangle$ has been neglected. The last subscript α of ${}^n\Psi_{(m)\alpha}$, where for the highest weight (16) $\alpha = j$, is the eigenvalue of $\Delta(s_z)$.

The different Young patterns describe the inequivalent irreducible representations, so the corresponding energies are generally different. However, for the same Young pattern $(N-n+1, n)$ there are M_n Young operators ${}^nC_{(m)}Z_n^N$ and M_n spaces with the same irreducible representation [5].

$$ M_n = \binom{N}{n} - \binom{N}{n-2}. \tag{17} $$

The eigenstates of the Hamiltonian are the combinations of ${}^n\Psi_{(m)j}$ with the same n:

$$ \begin{aligned} &H\ {}^n\Phi_{hj} = {}^nE_h\ {}^n\Phi_{hj} \\ &{}^n\Phi_{hj} = \sum_{(m)} a^h_{(m)}\ {}^n\Psi_{(m)j} \\ &h = 1, 2, \ldots, M_n. \end{aligned} \tag{18} $$

The partners ${}^n\Phi_{h\alpha}$ in the representation can be easily calculated by the lowering operator $\Delta(s_-)$ [7]:

$$ {}^n\Phi_{h\alpha} = [j-\alpha]^{-1}\Delta(s_-)\ {}^n\Phi_{h(\alpha+1)}. \tag{19} $$

It is straightforward to prove the following multiplication rules from the algebraic relations of the Temperley Lieb algebra:

(i) If $m_i - 1 > m_{i-1}$ or $i = 1$, we have

$$ e_{m_i}\ {}^nC_{(m)}Z_n^N = [2]\ {}^nC_{(m)}Z_n^N. \tag{20a} $$

(ii) If $m_i - 1 > m_{i-1}$ or $i = 1$, and $m_i > 2i - 1$, we have

$$ e_{m_i}\ {}^nC_{(m')}Z_n^N = {}^nC_{(m)}Z_n^N \tag{20b} $$

where

$$ \begin{aligned} &m'_p = m_p \qquad p \neq i \\ &m'_i = m_i - 1. \end{aligned} \tag{20c} $$

(iii) If $m_{i-1}+1<m_i<m_{i+1}-1$, $1<i<n$, or $i=1$, $m_1<m_2-1$, or $i=n$, $m_{n-1}+1<m_n<N$, we have (20*b*) where

$$\begin{aligned} &m'_p=m_p \qquad p\neq i \\ &m'_i=m_i+1. \end{aligned} \tag{20d}$$

(iv) If $m_{i-j}=m_i-2j$, $m_{i-1}=m_i-2$, $m_{i-l}\geq m_i-2l$, $l=2,3,\ldots,(j-1)$, we have (20*b*) where

$$\begin{aligned} &m'_p=m_p \qquad \text{when } p\geq i \text{ or } p<i-j \\ &m'_{i-1}=m_{i-1}+1 \\ &m'_{i-l}=m_{i-l+1} \qquad l=2,3,\ldots,j. \end{aligned} \tag{20e}$$

(v) If $m_i-1>m_{i-1}$ (or $i=1$), $m_{i+j}+1<m_{i+j+1}$ ($m_n<N$ when $i+j=n$), $m_i+2j=m_{i+j}$, and $m_i+2l\leq m_{i+l}$, $l=1,2,\ldots,(j-1)$, we have (20*b*) where

$$\begin{aligned} &m'_p=m_p \qquad \text{when } p>i+j \text{ or } p<i \\ &m'_{i+l-1}=m_{i+l} \qquad l=1,2,\ldots,j \\ &m'_{i+j}=m_{i+j}+1. \end{aligned} \tag{20f}$$

In terms of these rules, (18) becomes a set of M_n coupled linear algebraic equations with respect to $a^h_{(m)}$. We assume that $a^h_{(m)}=0$ if the following conditions are violated:

$$\begin{aligned} &1\leq m_1<m_2<\ldots<m_n\leq N \\ &m_i\geq 2i-1. \end{aligned} \tag{21}$$

For each set of $m_1, m_2, \ldots, m_n$, satisfying the conditions (21), we have an equation. On the right-hand side of the equation we have ${}^nE_h a^h_{(m)}$, and on its left-hand side there are the following terms:

(i) If there are l m_i satisfying $m_i-1=m_{i-1}$, $1<i\leq n$, we have a term $(n-l)[2]a^h_{(m)}$.

(ii) If $m_i-1>m_{i-1}$, or $i=1$, and $m_i>2i-1$, we have a term $a^h_{m_1\ \ldots\ m_{i-1}(m_i-1)m_{i+1}\ \ldots\ m_n}$.

(iii) If $m_{i-1}+1<m_i<m_{i+1}-1$, $1<i<n$, or $i=1$, $m_1<m_2-1$, or $i=n$, $m_{n-1}+1<m_n<N$, we have a term $a^h_{m_1\ \ldots\ m_{i-1}(m_i+1)m_{i+1}\ \ldots\ m_n}$.

(iv) If $m_{i-j}=m_i-2j$, $m_{i-1}=m_i-2$, $m_{i-l}\geq m_i-2l$, $l=2,3,\ldots,(j-1)$, we have a term $a^h_{(m')}$ where

$$\begin{aligned} &m'_p=m_p \qquad \text{when } p<i-j \text{ or } p\geq i \\ &m'_{i-1}=m_i-1 \\ &m'_{i-l}=m_{i-l+1} \qquad l=2,3,\ldots,j. \end{aligned}$$

(v) If $m_i-1>m_{i-1}$ (or $i=1$), $m_{i+j}+1<m_{i+j+1}$ ($m_n<N$ when $i+j=n$), $m_i+2j=m_{i+j}$, $m_i+2l\leq m_{i+l}$, $l=1,2,\ldots,(j-1)$, we have a term $a^h_{(m')}$ where

$$\begin{aligned} &m'_p=m_p \qquad \text{when } p<i \text{ or } p>i+j \\ &m'_{i+l-1}=m_{i+l} \qquad l=1,2,\ldots,j \\ &m'_{i+j}=m_{i+j}+1. \end{aligned}$$

Now, we give two examples to show how to write the linear algebraic equations. For the case that $N=10$, $n=5$ and $(m)=(1\,4\,5\,7\,9)$, we have $4[2]a^h_{14579}$ from rule (i),

a^h_{13579} from rule (ii), $a^h_{24579}+a^h_{14589}+a^h_{14570}$ from rule (iii), $a^h_{14679}+a^h_{14589}+a^h_{45679}+a^h_{45789}+a^h_{14789}$ from rule (iv), and $a^h_{45679}+a^h_{45789}+a^h_{45790}+a^h_{14590}$ from rule (v), where 0 denotes 10. Altogether, we obtain the equation for this case as follows:

$$4[2]a^h_{14579}+a^h_{13579}+a^h_{24579}+2a^h_{14589}+a^h_{14570}+a^h_{14679}+2a^h_{45679}+2a^h_{45789}$$
$$+a^h_{14789}+a^h_{45790}+a^h_{14590}={}^nE_h a^h_{14579}$$

Similarly, for the case that $N=10$, $n=5$ and $(m)=(2\,3\,6\,8\,0)$, we have $4[2]a^h_{23680}$ from rule (i), $a^h_{13680}+a^h_{23580}+a^h_{23670}+a^h_{23689}$ from rule (ii), $a^h_{23780}+a^h_{23690}$ from rule (iii), $a^h_{23780}+a^h_{23890}+a^h_{23690}$ from rule (iv), and a^h_{23890} from rule (v). Altogether, we obtain the equation as follows:

$$4[2]a^h_{23680}+a^h_{13680}+a^h_{23580}+a^h_{23670}+a^h_{23689}+2a^h_{23780}+2a^h_{23690}+2a^h_{23890}={}^nE_h a^h_{23680}.$$

In terms of the rules, the Hamiltonian equation becomes a set of coupled linear algebraic equations. It is a typical eigenvalue problem of a matrix with finite dimensions, so the problem can be easily solved by computer.

4. Examples: $n=1$ and $n=2$

In this section we are going to illustrate this method and compare it with the Bethe ansatz in terms of two simple examples, where $n=1$ and $n=2$.

For $n=0$, only one state $|\rangle_0$ with all spins up is concerned. It is obviously the eigenstate both of the Hamiltonian with zero energy and of the angular momentum with $j=(N+1)/2$.

For $n=1$, there are $N+1$ independent states. One of them is obtained from $|\rangle_0$ by the action of the lowering operator $\Delta(s_-)$:

$$\begin{aligned}&{}^0\Phi_{(N+1)/2}=|\rangle_0\\&{}^0\Phi_{(N-1)/2}=\Delta(s_-)|\rangle_0.\end{aligned}\tag{22}$$

The rest of the states with one down spin are those of the highest weights of the representations, whose Young pattern has only one box in the second row. Now, the Hamiltonian equation becomes very easy:

$$\begin{aligned}&{}^1\Phi_{h(N-1)/2}=\sum_m a^h_m\,{}^1C_m|1\rangle\\&([2]-{}^1E_h)a^h_m+a^h_{m+1}+a^h_{m-1}=0\\&a^h_0=a^h_{N+1}=0.\end{aligned}\tag{23}$$

It is easy to prove by induction that

$$\frac{a^h_m}{a^h_1}=(-1)^{m-1}\frac{\sin\{mh\pi/(N+1)\}}{\sin\{h\pi/(N+1)\}}=(-1)^{m-1}\sum_{k=0}^{m-1}e^{i(2k-m+1)h\pi/(N+1)}.\tag{24}$$

For simplicity we can put $a^h_1=1$. The energy is calculated from the condition $a^h_{N+1}=0$,

$$\begin{aligned}&{}^1E_h=[2]-2\cos\frac{h\pi}{N+1}\\&h=1,2,\ldots,N.\end{aligned}\tag{25}$$

In order to compare the results with those obtained from the Bethe ansatz, we sketch the latter [1, 4] as follows:

$$\begin{aligned} &H|n, h\rangle = {}^nE_h|n, h\rangle \\ &|n, h\rangle = \sum f_h(x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle \end{aligned} \tag{26}$$

where the summed x_i denotes the location of the ith down spin and $1 \leq x_1 < x_2 < \ldots < x_n \leq N+1 (n \leq (N+1)/2)$. The Bethe ansatz is

$$f_h(x_1, \ldots, x_n) = \sum_p \varepsilon_p A(k_{p_1}, \ldots, k_{p_n})\, e^{i(k_{p_1}x_1 + \cdots + k_{p_n}x_n)} \tag{27}$$

where the sum extends over all permutations and negations of the impulsions k_j, and ε_p changes sign at each transformation. The coefficients A are given by

$$\begin{aligned} &A(k_1, \ldots, k_n) = \prod_{j=1}^{n} \beta(-k_j) \prod_{1 \leq j < l \leq n} B(-k_j, k_l)\, e^{-ik_l} \\ &\beta(k) = (1 - q\, e^{-ik})\, e^{i(N+2)k} \\ &B(k_1, k_2) = (1 - [2]\, e^{ik_2} + e^{i(k_1+k_2)})(1 - [2]\, e^{-ik_1} + e^{i(k_2-k_1)}) \end{aligned} \tag{28}$$

where the impulsions k_j satisfy the compatibility conditions

$$e^{i2(N+1)k_j} = \prod_{l=1, l\neq j}^{n} \frac{B(-k_j, k_l)}{B(k_j, k_l)} \qquad j = 1, 2, \ldots, n. \tag{29a}$$

There are a few sets of solutions k_j labelled by a parameter h. The energy ${}^nE_h$ can be expressed in terms of the impulsions k_j as

$${}^nE_h = n[2] - 2 \sum_{j=1}^{n} \cos k_j. \tag{29b}$$

In fact, the impulsions k_j are hard to obtain except for $n = 1$ where the right-hand side of (29a) becomes one, and the result is the same as (25). The coincidence is not surprising because the energy is normally degenerate. The relation between the two representations is also very clear. For $n = 1$,

$$\sum_x f_h(x)|x\rangle \propto \sum_m a_m^h\, {}^1C_m|1\rangle.$$

Thus,

$$\begin{aligned} f_h(x) &\propto (-1)^{x-1}(q a_x^h - a_{x-1}^h) \\ &= \frac{(-1)^{h+1}}{2i \sin k}\{(1 - q\, e^{ik})\, e^{-i(N+2)k}\, e^{ikx} - (1 - q\, e^{-ik})\, e^{i(N+2)k}\, e^{-ikx}\}. \end{aligned} \tag{30}$$

The two results coincide with each other up to a factor of $(-1)^{h+1}(2i \sin k)^{-1}$.

The advantage of using the Bethe ansatz is that the general forms of the energy and the eigenstates have been obtained. Unfortunately, the compatibility condition (29a) is hard to solve analytically even for $n = 2$, so that we do not have explicit forms of the solutions. In addition, the form (27) cannot be used for cases where two or more impulsions coincide with each other, so this method is not convenient enough for the Beraha q-values (see next section).

Now, we turn to the case of $n = 2$. It is not expected that general forms of the solutions will be obtained because the compatibility condition (29a) does not have analytic solutions. However, for lower N (for instance, $N = 3$) we can obtain explicit solutions by using quantum Young operators.

According to the rules given in the previous section, for $n=2$, the Hamiltonian equation is as follows.

$$\begin{aligned} &H\,{}^2\Phi_{h(N-3)/2} = {}^2E_h\,{}^2\Phi_{h(N-3)/2} \\ &{}^2\Phi_{h(N-3)/2} = \sum_{m_1m_2} a^h_{m_1m_2}\,{}^2C_{m_1m_2}|1\,3\rangle \end{aligned} \tag{31}$$

and $a^h_{m_1m_2}$ satisfies the recursive relations

$$\begin{aligned} &(2[2]-{}^2E_h)a^h_{m(m+2)} + a^h_{(m-1)(m+2)} + a^h_{m(m+1)} + 2a^h_{(m+1)(m+2)} + a^h_{m(m+3)} + a^h_{(m+2)(m+3)} = 0 \\ &([2]-{}^2E_h)a^h_{m(m+1)} + a^h_{(m-1)(m+1)} + a^h_{m(m+2)} = 0 \\ &(2[2]-{}^2E_h)a^h_{m_1m_2} + a^h_{(m_1-1)m_2} + a^h_{m_1(m_2-1)} + a^h_{(m_1+1)m_2} + a^h_{m_1(m_2+1)} = 0 \\ &\qquad \text{when } m_1+2<m_2 \end{aligned} \tag{32}$$

where $a^h_{m_1m_2}=0$ if condition (21) is violated. Letting $a^h_{13}=1$, we can obtain all the $a^h_{m_1m_2}$ and energy ${}^2E_h$ by solving (32). For example, for $N=3$, we have

$$\begin{aligned} &(2[2]-{}^2E_h)a^h_{13} + 2a^h_{23} = 0 \\ &([2]-{}^2E_h)a^h_{23} + a^h_{13} = 0. \end{aligned}$$

Thus,

$$\begin{aligned} &{}^2E_h = 2[2] - \tfrac{1}{2}([2] \mp \sqrt{[3]+9}) \\ &a^h_{13} = 1 \qquad a^h_{23} = \tfrac{1}{4}(-[2] \pm \sqrt{[3]+9}). \end{aligned} \tag{33}$$

5. Beraha values

We have presented a method to compute the eigenstates and energies for the *XXZ* spin chain model with the symmetry of $U_q\,\mathrm{sl}(2)$. The results hold for any q. But for the generic q-values, the energies are normally degenerate, and the eigenstates with different energies or different angular momenta are orthogonal to each other, where the inner product of two states are defined to be bilinear for both states, and

$$\langle x_1, \ldots, x_n | x'_1, \ldots, x'_{n'} \rangle = \delta_{nn'}\delta_{x_1x'_1} \ldots \delta_{x_nx'_n}. \tag{34}$$

When q is a root of unity (the Beraha value), some energies may be accidentally degenerate, and the corresponding eigenstates may coincide with each other up to a coefficient. Consider the limit process that a generic q-value tends to a Beraha q-value. Two eigenstates remain orthogonal in the limit process so that the coincidental eigenstate must have a zero norm. On the other hand, if one of the coincidental states is, as it usually is, a highest weight before coincidence, the state after coincidence must also be a highest weight, i.e. it is annihilated by $\Delta(s_+)$.

Since two states coincide with each other, the representation becomes reducible, and another independent state is needed to complement the invariant representation space. The new state can be obtained from the difference of the two coincidental states by a limit process.

Let us discuss this in some detail. Suppose there are two highest-weight states, ${}^{n}\Phi_{hj}$ and ${}^{n'}\Phi_{n'j'}$, where $j=(N-2n+1)/2$, $j'=(N-2n'+1)/2$ and $m=n-n'>0$. For the generic q-value we have

$$\begin{aligned}
&H\,{}^{n}\Phi_{hj}={}^{n}E_{h}\,{}^{n}\Phi_{hj} \qquad H\,{}^{n'}\Phi_{h'j'}={}^{n'}E_{h}\,{}^{n'}\Phi_{h'j'}\\
&({}^{n}\Phi_{hj},{}^{n'}\Phi_{hj})=0\\
&\Delta(s_{+})\,{}^{n}\Phi_{hj}=0 \qquad \Delta(s_{+})\,{}^{n'}\Phi_{h'j}=0\\
&{}^{n'}\Phi_{h'j}=\left(\prod_{l=1}^{m}[l]^{-1}\Delta(s_{-})\right){}^{n'}\Phi_{h'j'}.
\end{aligned} \tag{35}$$

We assume that when q tends to the Beraha value $q_p=\exp(\mathrm{i}\pi p'/p)$, were p and p' are positive integers without a common factor, $p>p'$, $p>m$, two energies coincide to each other and

$$\lim_{q\to q_p}{}^{n'}\Phi_{h'j}=c\,{}^{n}\Phi_{hj}. \tag{36}$$

In the limit process that q goes to q_p, (35) holds so that when $q=q_p$ we have

$$\begin{aligned}
&({}^{n'}\Phi_{h'j},{}^{n'}\Phi_{h'j})=0\\
&\Delta(s_{+})\,{}^{n'}\Phi_{h'j}=0
\end{aligned} \tag{37}$$

and all the states $\Delta(s_{-})^{l}\,{}^{n'}\Phi_{h'j'}$, $m\leq l<p$ have the zero norm. On the other hand, another independent state can be obtained from the difference of two states by the following limit process:

$${}^{n}\Phi'_{hj}=\lim_{q\to q_p}\frac{{}^{n'}\Phi_{h'j}-c\,{}^{n}\Phi_{hj}}{q-q_p} \tag{38}$$

where ${}^{n}\Phi'_{hj}$ is independent of ${}^{n'}\Phi_{h'j}$, and not orthogonal to it. Under the action of $\Delta(s_{+})$, ${}^{n}\Phi'_{hj}$ becomes proportional to the state ${}^{n'}\Phi_{h'(j+1)}$. From the states ${}^{n'}\Phi_{h'j'}$ and ${}^{n}\Phi'_{hj}$ we can obtain all the states which span a reducible but indecomposable representation (type I) by the lowering operators $\Delta(s_{-})$ and $\Delta(s_{-})^{p}/[p]!$ The picture for the type I representation was described by Pasquier and Saleur [4], but they did not discuss how to obtain the new states.

Now, we are going to discuss the condition for appearance of type I representations, which is the same as that of coincidence of two energies when q tends to q_p. For the Beraha q-values, the energy formula (29) still holds except that the right-hand side of (29a) sometimes becomes indeterminate. Defining a rule for determining the indeterminate forms, we obtain the condition for appearance of type I representations.

Firstly, we assume that one of the impulsions k_j in (29) is equal to $\gamma=-\mathrm{i}\log q$; for definiteness, $k_i=\gamma$. Because

$$\frac{B(-k_j,\gamma)}{B(k_j,\gamma)}=1 \qquad \frac{B(-\gamma,k_j)}{B(\gamma,k_j)}=q^{4}. \tag{39}$$

The compatibility condition (29a) for n with $j\neq i$ is the same as that for $n'=n-1$, and the condition with $j=i$ is a constraint for q,

$$q^{2(N+1)}=q^{4(n-1)}$$

that is,

$$[p]=0 \qquad n+n'=N+2 \bmod p. \tag{40a}$$

Since $2\cos k_i=[2]$, (29*b*) becomes

$$^{n}E_h={}^{n'}E_{h'} \qquad n'=n-1. \tag{40b}$$

The rest of the impulsions k_j for $^{n}E_h$ are the same as those for $^{n'}E_h$. As pointed out by Pasquier and Saleur, when $k_i=\gamma$, there is no reflection wave. However, it is easy to check that when one impulsion $k_i=\gamma$, the eigenstate $|n,h\rangle$ is proportional to $\Delta(s_-)|n',h'\rangle$.

Now, we discuss the cases where more than one impulsion is equal to γ: $k_{i_1}=k_{i_2}=\ldots=k_{i_m}=\gamma$. In these cases the indeterminate factor $B(-\gamma,\gamma)/B(\gamma,\gamma)$ appears in (29*a*). From (39) we know that

$$\lim_{k_1\to\gamma}\lim_{k_2\to\gamma}\frac{B(-k_1,k_2)}{B(k_1,k_2)}=1$$

$$\lim_{k_2\to\gamma}\lim_{k_1\to\gamma}\frac{B(-k_1,k_2)}{B(k_1,k_2)}=q^4.$$

If we define

$$\lim_{k_1,k_2\to\gamma}\frac{B(-k_1,k_2)}{B(k_1,k_2)}=q^2 \tag{41}$$

the compatibility condition (29*a*) for n with $k_j\neq\gamma$ becomes the same as that for $n'=n-m$ and the condition with $k_j=\gamma$ becomes a constraint for q:

$$q^{2(N+1)}=q^{4(n-m)}q^{2(m-1)}.$$

That is,

$$[p]=0 \qquad n+n'=N+2 \bmod p \tag{42}$$

and (29*b*) becomes

$$^{n'}E_{h'}={}^{n}E_h \qquad n'=n-m. \tag{43}$$

However, the eigenstates (27) are vanishing in these cases. Extracting the zero factors carefully, one will be able to obtain the right eigenstates, which should coincide with those obtained by using quantum Young operators up to a coefficient.

The condition (42) for the appearance of the reducible but indecomposable representations is the same as that given by Levy [5] and Pasquier-Saleur [4].

The definition for the limit (41) can be equivalently realized by the following limit:

$$\begin{aligned} &k_{i_1}=k_{i_2}=\ldots=k_{i_m}=\gamma+\varepsilon && \text{if } m \text{ is odd}\\ &k_{i_1}=k_{i_2}=\ldots=k_{i_{m-1}}=\gamma+\varepsilon \qquad k_{i_m}=\gamma-\varepsilon && \text{if } m \text{ is even}\end{aligned} \tag{44}$$

and $\varepsilon\to 0$.

Finally, we are going to discuss a simple example where $N=3$. From (17) we have one quintuplet ($n=0, j=2$), three triplets ($n=1, j=1$) and two singlets ($n=2, j=0$).

The energies and the eigenstates of the highest weights are obtained from (22), (24), (25) and (33) as follows:

$$ {}^0E=0 $$
$$ {}^0\Phi_2=|\,\rangle_0 $$
$$ {}^1E_1=[2]-\sqrt{2} $$
$$ {}^1\Phi_{11}=({}^1C_1-\sqrt{2}\,{}^1C_2+{}^1C_3)|1\rangle $$
$$ =q|1\rangle+(\sqrt{2}q-1)|2\rangle+(q-\sqrt{2})|3\rangle-|4\rangle $$
$$ {}^1E_2=[2] $$
$$ {}^1\Phi_{21}=({}^1C_1-{}^1C_3)|1\rangle $$
$$ =q|1\rangle-|2\rangle-q|3\rangle+|4\rangle $$
$$ {}^1E_3=[2]+\sqrt{2} $$
$$ {}^1\Phi_{31}=({}^1C_1+\sqrt{2}\,{}^1C_2+{}^1C_3)|1\rangle $$
$$ =q|1\rangle-(\sqrt{2}q+1)|2\rangle+(q+\sqrt{2})|3\rangle-|4\rangle $$
$$ {}^2E_1=\tfrac{1}{2}(3[2]+\sqrt{[3]+9}) $$
$$ {}^2\Phi_{10}=\{{}^2C_{13}-\tfrac{1}{4}([2]-\sqrt{[3]+9})\,{}^2C_{23}\}|1\,3\rangle $$
$$ =\tfrac{1}{4}\{q^2([2]-\sqrt{[3]+9})|1\,2\rangle+(3q^2-1+q\sqrt{[3]+9})|1\,3\rangle-4q(|1\,4\rangle $$
$$ +|2\,3\rangle)+(3-q^2+q\sqrt{[3]+9})|2\,4\rangle+([2]-\sqrt{[3]+9}|3\,4\rangle\} $$
$$ {}^2E_2=\tfrac{1}{2}(3[2]-\sqrt{[3]+9}) $$
$$ {}^2\Phi_{20}=\{{}^2C_{13}-\tfrac{1}{4}([2]+\sqrt{[3]+9})\,{}^2C_{23}\}|1\,3\rangle $$
$$ =\tfrac{1}{4}\{q^2([2]+\sqrt{[3]+9})|1\,2\rangle+(3q^2-1-q\sqrt{[3]+9})|1\,3\rangle-4q(|1\,4\rangle $$
$$ +|2\,3\rangle)+(3-q^2-q\sqrt{[3]+9})|2\,4\rangle+([2]+\sqrt{[3]+9})|3\,4\rangle\}. $$

Now, we discuss some cases where q is a root of unity and the condition (42) is satisfied:

(i) When $p=2$, $q_p=\exp(i\pi/2)=i$ and $[2]=0$, we have

$$ {}^0E={}^1E_2=0 $$
$$ {}^1E_1={}^2E_2=-\sqrt{2} $$
$$ {}^1E_3={}^2E_1=\sqrt{2} $$

and

$$ {}^0\Phi_1=\Delta(s_-)\,{}^0\Phi_2=e^{i3\pi/4}\,{}^1\Phi_{21} $$
$$ =e^{-i\pi/4}(-i|1\rangle+|2\rangle+i|3\rangle-|4\rangle) $$
$$ {}^1\Phi_{10}=\Delta(s_-)\,{}^1\Phi_{11}=-2\,e^{i\pi/4}\,{}^2\Phi_{20} $$
$$ =e^{i\pi/4}\{\sqrt{2}|1\,2\rangle+(2+i\sqrt{2})|1\,3\rangle+2i(|1\,4\rangle $$
$$ +|2\,3\rangle)-(2-i\sqrt{2})|2\,4\rangle-\sqrt{2}|3\,4\rangle\} $$
$$ {}^1\Phi_{30}=\Delta(s_-)\,{}^1\Phi_{31}=-2\,e^{i\pi/4}\,{}^2\Phi_{10} $$
$$ =e^{i\pi/4}\{-\sqrt{2}|1\,2\rangle+(2-i\sqrt{2})|1\,3\rangle+2i(|1\,4\rangle $$
$$ +|2\,3\rangle)-(2+i\sqrt{2})|2\,4\rangle+\sqrt{2}|3\,4\rangle\}. $$

They all have zero norm and are annihilated by $\Delta(s_+)$. The new states are

$$
\begin{aligned}
{}^1\Phi'_{21} &= \lim_{q\to q_p} \frac{{}^0\Phi_1 - e^{i3\pi/4}\,{}^1\Phi_{21}}{q-q_p} \\
&= \tfrac{1}{2}\, e^{-i\pi/4}(5|1\rangle + i|2\rangle - |3\rangle + 3i|4\rangle) \\
{}^2\Phi'_{20} &= \lim_{q\to q_p} \frac{{}^1\Phi_{10} + 2\,e^{i\pi/4}\,{}^2\Phi_{20}}{q-q_p} \\
&= \frac{1}{\sqrt{2}} e^{-i\pi/4}\{-3|1\,2\rangle - (3\sqrt{2}+i)|1\,3\rangle - (2+i\sqrt{2})(|1\,4\rangle \\
&\quad + |2\,3\rangle) - (\sqrt{2}+i)|2\,4\rangle - |3\,4\rangle\} \\
{}^2\Phi'_{10} &= \lim_{q\to q_p} \frac{{}^1\Phi_{30} + 2\,e^{i\pi/4}\,{}^2\Phi_{10}}{q-q_p} \\
&= \frac{1}{\sqrt{2}} e^{-i\pi/4}\{3|1\,2\rangle - (3\sqrt{2}-i)|1\,3\rangle + (2-i\sqrt{2})(|1\,4\rangle \\
&\quad + |2\,3\rangle) - (\sqrt{2}-i)|2\,4\rangle + |3\,4\rangle\} \\
&\Delta(s_+)\,{}^1\Phi'_{21} = -4\,{}^0\Phi_2 \\
&\Delta(s_+)\,{}^2\Phi'_{20} = 2\,{}^1\Phi_{11} \\
&\Delta(s_+)\,{}^2\Phi'_{10} = 2\,{}^1\Phi_{31}.
\end{aligned}
$$

Under the action of $\Delta(s_-)$ and $\Delta(s_-)^2/[2]$ we obtain three spaces of the type I representations with the dimensions 8, 4 and 4, respectively.

(ii) When $p=3$, $p'=1$, $q_p=\exp(i\pi/3)$ and $[3]=0$, we have

$$
\begin{aligned}
&{}^0E = {}^2E_2 = 0 \\
&{}^0\Phi_0 = [2]^{-1}\Delta(s_-)^2\,{}^0\Phi_2 = -e^{-i\pi/3}\,{}^2\Phi_{20} \\
&\quad = e^{-i2\pi/3}|1\,2\rangle + e^{-i\pi/3}|1\,3\rangle + |1\,4\rangle + |2\,3\rangle + e^{i\pi/3}|2\,4\rangle + e^{i2\pi/3}|3\,4\rangle
\end{aligned}
$$

and ${}^0\Phi_0$ has zero norm and is annihilated by $\Delta(s_+)$. The new state is

$$
\begin{aligned}
{}^2\Phi'_{20} &= \lim_{q\to q_p} \frac{{}^0\Phi_0 + e^{-i\pi/3}\,{}^2\Phi_{20}}{q-q_p} \\
&= \tfrac{1}{6}\{(24+i2\sqrt{3})|1\,2\rangle + (9+i5\sqrt{3})|1\,3\rangle - (3-i3\sqrt{3})(|1\,4\rangle \\
&\quad + |2\,3\rangle) + i2\sqrt{3}|2\,4\rangle + (9+i5\sqrt{3})|3\,4\rangle\} \\
&\Delta(s_+)^2\Phi'_{20} = 2\sqrt{3}\,e^{i\pi/6}\,{}^0\Phi_1.
\end{aligned}
$$

Under the action of $\Delta(s_-)$ and $\Delta(s_-)^3/[3]!$, ${}^0\Phi_2$ and ${}^2\Phi'_{20}$ generate a space of the type I representation with the dimension 6.

(iii) When $p=3$, $p'=2$, $q_p=\exp(i2\pi/3)$ and $[3]=0$, we have

$$
\begin{aligned}
&{}^0E = {}^2E_1 = 0 \\
&{}^0\Phi_0 = [2]^{-1}\Delta(s_-)^2\,{}^0\Phi_2 = -e^{-i2\pi/3}\,{}^2\Phi_{10} \\
&\quad = -e^{-i\pi/3}|1\,2\rangle - e^{i\pi/3}|1\,3\rangle + |1\,4\rangle + |2\,3\rangle - e^{-i\pi/3}|2\,4\rangle - e^{i\pi/3}|3\,4\rangle
\end{aligned}
$$

and ${}^0\Phi_0$ has zero norm and is annihilated by $\Delta(s_+)$. The new state is

$$
\begin{aligned}
{}^2\Phi'_{10} = \lim_{q\to q_p} \frac{{}^0\Phi_0 + \mathrm{e}^{-i2\pi/3}\,{}^2\Phi_{10}}{q-q_p} \\
&= \tfrac{1}{6}\{(-24+i2\sqrt{3})|1\,2\rangle + (9-i5\sqrt{3})|1\,3\rangle + (3+i3\sqrt{3})(|1\,4\rangle \\
&\quad + |2\,3\rangle) - i2\sqrt{3}|2\,4\rangle - (9-i5\sqrt{3})|3\,4\rangle\} \\
\Delta(s_+)\,{}^2\Phi'_{10} &= -2\sqrt{3}\,\mathrm{e}^{-i\pi/6}\,{}^0\Phi_1 .
\end{aligned}
$$

Under the action of $\Delta(s_-)$ and $\Delta(s_-)^3/[3]!$, ${}^0\Phi_2$ and ${}^2\Phi'_{10}$ generate a space of the type I representation with the dimension 6.

(iv) When $p=4$, $p'=1$, $q_p=\exp(i\pi/4)$ and $[4]=0$, we have

$$
\begin{aligned}
{}^0E &= {}^1E_1 = 0 \\
{}^0\Phi_1 &= \Delta(s_-)\,{}^0\Phi_2 = \mathrm{e}^{-i5\pi/8}\,{}^1\Phi_{11} \\
&= \mathrm{e}^{i\pi/8}(-i|1\rangle + \mathrm{e}^{-i\pi/4}|2\rangle + |3\rangle + \mathrm{e}^{i\pi/4}|4\rangle)
\end{aligned}
$$

and ${}^0\Phi_1$ has zero norm and is annihilated by $\Delta(s_+)$. The new state is

$$
\begin{aligned}
{}^1\Phi'_{11} &= \lim_{q\to q_p} \frac{{}^0\Phi_1 - \mathrm{e}^{-i5\pi/8}\,{}^1\Phi_{11}}{q-q_p} \\
&= \tfrac{1}{4}\, e^{-i\pi/8}\{10i|1\rangle - \sqrt{2}(1-5i)|2\rangle + (2+4i)|3\rangle + 3\sqrt{2}\,(1+i)|4\rangle\} \\
\Delta(s_+)\,{}^1\Phi'_{11} &= 4\sqrt{2}\,\mathrm{e}^{i\pi/4}\,{}^0\Phi_2
\end{aligned}
$$

Under the action of $\Delta(s_-)$ and $\Delta(s_-)^4/[4]!$, ${}^0\Phi_1$ and ${}^1\Phi'_{11}$ generate a space of the type I representation with the dimension 8.

(v) When $p=4$, $p'=3$, $q_p=\exp(i3\pi/4)$ and $[4]=0$, we have

$$
\begin{aligned}
{}^0E &= {}^1E_3 = 0 \\
{}^0\Phi_1 &= \Delta(s_-)\,{}^0\Phi_2 = \mathrm{e}^{i\pi/8}\,{}^1\Phi_{31} \\
&= \mathrm{e}^{-i\pi/8}(-|1\rangle + \mathrm{e}^{-i\pi/4}|2\rangle + i|3\rangle - \mathrm{e}^{i\pi/4}|4\rangle)
\end{aligned}
$$

and ${}^0\Phi_1$ has zero norm and is annihilated by $\Delta(s_+)$. The new state is

$$
\begin{aligned}
{}^1\Phi'_{31} &= \lim_{q\to q_p} \frac{{}^0\Phi_1 - \mathrm{e}^{i\pi/8}\,{}^1\Phi_{31}}{q-q_p} \\
&= \tfrac{1}{4}\,\mathrm{e}^{i\pi/8}\{-10|1\rangle + \sqrt{2}(5-i)|2\rangle - (4+2i)|3\rangle + 3\sqrt{2}(1+i)|4\rangle\} \\
\Delta(s_+)\,{}^1\Phi'_{31} &= 4\sqrt{2}\,\mathrm{e}^{-i\pi/4}\,{}^0\Phi_2 .
\end{aligned}
$$

Under the action of $\Delta(s_-)$ and $\Delta(s_-)^4/[4]!$, ${}^0\Phi_2$ and ${}^1\Phi'_{31}$ generate a space of the type I representation with the dimension 8.

Acknowledgments

Two of the authors (B-YH and ZQM) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy. The authors are indebted to the adjudicator who read and corrected our manuscript carefully. This work was partly supported by the Natural Science Foundation of China through the Nankai Institute of Mathematics.

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